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# Coefficient estimates for subclass of analytic and bi-univalent functions defined by differential operator

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**Abstract**

In this paper, we introduce and investigate a subclass $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. Upper bounds for the second and third coefficients of functions in this subclass are founded. Our results, which are presented in this paper, generalize and improve those in related works of several earlier authors.

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## 1 Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We also denote by $\mathcal{S}$ the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by $\Sigma$.
Determination of the bounds for the coefficients $a_n$ is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example,

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the bound for the second coefficient $a_2$ of functions $f \in \mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Recently some researchers have been devoted to study the bi-univalent functions class $\Sigma$ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class $\Sigma$, see [14]. In fact that this widely-cited work by Srivastava et al. [14] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al.[15, 16, 17, 18, 19, 20], and others [6, 9, 12]. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \Sigma$, is still an open problem. In fact there is no direct way to get bound for coefficients greater than three. In special cases there are some papers in which the Faber polynomial methods were used for determining upper bounds for higher-order coefficients (for example see [2, 17]).

More recently, Caglar [6] introduced the following two subclasses of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

**Definition 1.1.** (see [6]) Let $0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \geq 0$, a function $f(z)$ given by (1.1) is said to be in the class $N_{\Sigma}^{\mu}(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad and \qquad \left| arg\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}),$$

and

$$\left| arg\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}),$$

where the function $g$ is given by (1.2).

**Theorem 1.2.** *(see [6]) Let $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\mu}(\alpha, \lambda)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}}, \qquad |a_3| \leq \frac{2\alpha}{(2\lambda+\mu)} + \frac{4\alpha^2}{(\lambda+\mu)^2}.$$

**Definition 1.3.** (see [6]) Let $0 \leq \beta < 1,\ \lambda \geq 1,\ \mu \geq 0$, a function $f(z)$ given by (1.1) is said to be in the class $N_{\Sigma}^{\mu}(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad and \qquad \mathfrak{Re}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right] > \beta, \qquad (z \in \mathbb{U}),$$

and

$$\mathfrak{Re}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right] > \beta, \qquad (w \in \mathbb{U}),$$

where the function $g$ is given by (1.2).

**Theorem 1.4.** *(see [6]) Let $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\mu}(\beta,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{(\lambda+\mu)}, \sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}}\right\},$$

*and*

$$|a_3| \leq \begin{cases} \min\{\frac{2(1-\beta)}{(2\lambda+\mu)} + \frac{4(1-\beta)^2}{(\lambda+\mu)^2}, \frac{4(1-\beta)}{(1+\mu)(2\lambda+\mu)}\}, & 0 \leqslant \mu \ < 1 \\ \frac{2(1-\beta)}{(\mu+2\lambda)}, & \mu \geq 1. \end{cases}$$

As a generalization of two subclasses $N_{\Sigma}^{\mu}(\alpha,\lambda)$ and $N_{\Sigma}^{\mu}(\beta,\lambda)$, Bulut [4] introduced two subclasses $N_{\Sigma}^{\delta,\mu}(n,\alpha,\lambda)$ and $N_{\Sigma}^{\delta,\mu}(n,\alpha,\lambda)$ of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients.
The purpose of this paper is to investigate the bi-univalent function subclass $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$ and derive coefficient estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$. Our results for the bi-univalent function class $f \in N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$ would generalize and improve some recent works due to Brannan and Taha [3], Bulut [4], Caglar [6], Frasin and Aouf [9], Porwal and Darus [12] and Srivastava [14].

## 2 The subclass $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$

In this section, we introduce and investigate the general subclass $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$. For $f \in \mathcal{A}$, we consider the following differential operator which introduced by Al-Oboudi [1]:

$$D_{\delta}^{0}f(z) = f(z), \tag{2.1}$$
$$D_{\delta}^{1}f(z) = (1-\delta)f(z) + \delta z f'(z) \qquad (\delta \geq 0), \tag{2.2}$$
$$D_{\delta}^{n}f(z) = D_{\delta}(D_{\delta}^{n-1}f(z)) \qquad (n \in \mathbb{N}). \tag{2.3}$$

If $f$ is given by (1.1), then from (2.2) and (2.3) we see that

$$D_{\delta}^{n}f(z) = z + \sum_{k=2}^{\infty}[1+(k-1)\delta]^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}), \tag{2.4}$$

with $D_{\delta}^{n}f(0) = 0$.

**Definition 2.1.** Let $h, p : \mathbb{U} \to \mathbb{C}$ be analytic functions such that

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \quad (z \in \mathbb{U}) \quad \textit{and} \quad h(0) = p(0) = 1.$$

A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \textit{and} \quad \left[(1-\lambda)\left(\frac{D_{\delta}^{n}f(z)}{z}\right)^{\mu} + \lambda\,(D_{\delta}^{n}f(z))'\left(\frac{D_{\delta}^{n}f(z)}{z}\right)^{\mu-1}\right] \in h(\mathbb{U}), \tag{2.5}$$

and

$$\left[(1-\lambda)\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu}+\lambda\left(D_\delta^n g(w)\right)'\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu-1}\right]\in p(\mathbb{U}), \tag{2.6}$$

where the function $g$ is defined by (1.2).

**Remark 2.2.** There are many choices of $h$ and $p$ which would provide interesting subclasses of class $N_\Sigma^{h,p}(n,\delta,\mu,\lambda)$. For example, if we take

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0<\alpha\leq 1, \lambda\geq 1, z\in\mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the assumptions of Definition 2.1. If $f\in N_\Sigma^{h,p}(n,\delta,\mu,\lambda)$, then

$$f\in\Sigma \quad \textit{and} \qquad \left|arg\left[(1-\lambda)\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu}+\lambda\left(D_\delta^n f(z)\right)'\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu-1}\right]\right|<\frac{\alpha\pi}{2} \quad (z\in\mathbb{U}),$$

and

$$\left|arg\left[(1-\lambda)\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu}+\lambda\left(D_\delta^n g(w)\right)'\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu-1}\right]\right|<\frac{\alpha\pi}{2} \quad (w\in\mathbb{U}).$$

Therefore in this case, it reduce to class which defined by Bulut [4, Definition 2], and if we take $n=0$ it reduce to class in Definition 1.1.
If we take

$$h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z} \qquad (0\leq\beta<1, \lambda\geq 1, z\in\mathbb{U}),$$

then the functions $h(z)$ and $p(z)$ satisfy the assumptions of Definition 2.1. If $f\in N_\Sigma^{h,p}(n,\delta,\mu,\lambda)$, then

$$f\in\Sigma \quad \textit{and} \qquad \mathfrak{Re}\left[(1-\lambda)\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu}+\lambda\left(D_\delta^n f(z)\right)'\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu-1}\right]>\beta, \quad (z\in\mathbb{U}),$$

and

$$\mathfrak{Re}\left[(1-\lambda)\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu}+\lambda\left(D_\delta^n g(w)\right)'\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu-1}\right]>\beta, \quad (w\in\mathbb{U}).$$

Therefore in this case, it reduce to class which defined by Bulut [4, Definition 10], and if we take $n=0$ it reduce to class in Definition 1.3.

### 2.1 Coefficient Estimates

Now, we derive the estimates of the coefficients $|a_2|$ and $|a_3|$ for class $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$.

**Theorem 2.3.** *Let $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\delta)^{2n}(\lambda+\mu)^2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{2|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}}\right\}, \tag{2.7}$$

*and*

$$\begin{aligned}|a_3| \leq \min\{&\frac{|h''(0)|+|p''(0)|}{4(1+2\delta)^n(2\lambda+\mu)}+\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\delta)^{2n}(\lambda+\mu)^2},\\ &\frac{|h''(0)|[|4(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|]+|p''(0)|(1+\delta)^{2n}|\mu-1|}{4(1+2\delta)^n|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}\}.\end{aligned} \tag{2.8}$$

*Proof.* Since $f \in N_{\Sigma}^{h,p}(n,\delta,\mu,\lambda)$ and $g = f^{-1}$. Therefore from relations (2.5) and (2.6) we have

$$\left[(1-\lambda)\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu}+\lambda\,(D_\delta^n f(z))'\left(\frac{D_\delta^n f(z)}{z}\right)^{\mu-1}\right]=h(z) \quad (z\in\mathbb{U}), \tag{2.9}$$

and

$$\left[(1-\lambda)\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu}+\lambda\,(D_\delta^n g(w))'\left(\frac{D_\delta^n g(w)}{w}\right)^{\mu-1}\right]=p(w) \quad (w\in\mathbb{U}), \tag{2.10}$$

respectively, where functions $h$ and $p$ satisfy the conditions of Definition 2.1. Also, the functions $h$ and $p$ have the following Taylor-Maclaurin series expansions:

$$h(z)=1+h_1z+h_2z^2+h_3z^3+\cdots, \tag{2.11}$$

and

$$p(w)=1+p_1w+p_2w^2+p_3w^3+\cdots. \tag{2.12}$$

Now, by substituting (2.11) and (2.12) into (2.9) and (2.10), respectively, and equating the coefficients, we get

$$(1+\delta)^n(\lambda+\mu)a_2=h_1, \tag{2.13}$$

$$(1+2\delta)^n(2\lambda+\mu)a_3+(1+\delta)^{2n}(\mu-1)(\lambda+\frac{\mu}{2})a_2^2=h_2, \tag{2.14}$$

$$-(1+\delta)^n(\lambda+\mu)a_2=p_1, \tag{2.15}$$

and

$$-(1+2\delta)^n(2\lambda+\mu)a_3+[4(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)](\lambda+\frac{\mu}{2})a_2^2=p_2. \tag{2.16}$$

From (2.13) and (2.15), it yields

$$h_1 = -p_1, \tag{2.17}$$

and

$$2(1+\delta)^{2n}(\lambda+\mu)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.18}$$

Adding (2.14) and (2.16), it yields

$$[2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)](2\lambda+\mu)a_2^2 = p_2 + h_2. \tag{2.19}$$

Consequently, from (2.18) and (2.19), we have

$$a_2^2 = \frac{h_1^2 + p_1^2}{2(1+\delta)^{2n}(\lambda+\mu)^2}, \tag{2.20}$$

and

$$a_2^2 = \frac{p_2 + h_2}{[2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)](2\lambda+\mu)}, \tag{2.21}$$

respectively. Hence, from equations (2.20) and (2.21), we find that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\delta)^{2n}(\lambda+\mu)^2},$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{2|2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)},$$

Thus, the desired estimate on the coefficient $|a_2|$ as asserted in (2.7).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (2.16) from (2.14), we get

$$2(1+2\delta)^n(2\lambda+\mu)a_3 - 2(1+2\delta)^n(2\lambda+\mu)a_2^2 = h_2 - p_2. \tag{2.22}$$

Upon substituting the value of $a_2^2$ from (2.20) into (2.22), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2(1+\delta)^{2n}(\lambda+\mu)^2} + \frac{h_2 - p_2}{2(1+2\delta)^n(2\lambda+\mu)},$$

giving rise to

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\delta)^{2n}(\lambda+\mu)^2} + \frac{|h''(0)| + |p''(0)|}{4(1+2\delta)^n(2\lambda+\mu)}, \tag{2.23}$$

On the other hand, by substituting the value of $a_2^2$ from (2.21) into (2.22), it follows that

$$a_3 = \frac{(p_2 + h_2)}{[2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)](2\lambda+\mu)} + \frac{(h_2 - p_2)}{2(1+2\delta)^n(2\lambda+\mu)},$$

Hence

$$|a_3| \leq \frac{|h''(0)|\left(\left|4(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)\right|\right)+|p''(0)|(1+\delta)^{2n}|\mu-1|}{4(1+2\delta)^n|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}. \tag{2.24}$$

The desired estimate of the coefficient $|a_3|$ as asserted in (2.8) will be obtained from (2.23) and (2.24). This completes the proof. Q.E.D.

## 3 Conclusions

By choosing

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0<\alpha\leq 1,\ z\in\mathbb{U}),$$

in Theorem 2.3, we have the following result.

**Corollary 3.1.** *Let the function $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\delta,\mu}(n,\alpha,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2\alpha}{(1+\delta)^n(\lambda+\mu)}, \frac{2\alpha}{\sqrt{|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(\mu+2\lambda)}}\right\},$$

*and*

$$|a_3| \leq \min\{\frac{2\alpha^2}{(1+2\delta)^n(2\lambda+\mu)}+\frac{4\alpha^2}{(1+\delta)^{2n}(\lambda+\mu)^2}, \frac{\alpha^2[|4(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|+(1+\delta)^{2n}|\mu-1|]}{(1+2\delta)^n|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}\}.$$

**Remark 3.2.** It is easy to see, for the coefficient $|a_3|$, that

$$\frac{2\alpha^2}{(1+2\delta)^n(2\lambda+\mu)}+\frac{4\alpha^2}{(1+\delta)^{2n}(\lambda+\mu)^2} \leq \frac{2\alpha}{(1+2\delta)^n(2\lambda+\mu)}+\frac{4\alpha^2}{(1+\delta)^{2n}(\lambda+\mu)^2}.$$

Therefore Corollary 3.1 is an improvement of a result which obtained by Bulut [4, Theorem 4].

If we take $n=0$ in Corollary 3.1, then we have

**Corollary 3.3.** *Let the function $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\mu}(\alpha,\lambda)$. Then*

$$|a_2| \leq \begin{cases} \dfrac{2\alpha}{\sqrt{(\mu+1)(2\lambda+\mu)}}, & 1\leq\lambda\ <1+\sqrt{1+\mu} \\[2ex] \dfrac{2\alpha}{\lambda+\mu}, & \lambda\geq 1+\sqrt{1+\mu} \end{cases}$$

*and*

$$|a_3| \leq \begin{cases} \min\{\frac{2\alpha^2}{(2\lambda+\mu)}+\frac{4\alpha^2}{(\lambda+\mu)^2}, \frac{4\alpha^2}{(1+\mu)(2\lambda+\mu)}\}, & 0\leqslant\mu\ <1 \\ \frac{2\alpha^2}{(\mu+2\lambda)}, & \mu\geq 1. \end{cases}$$

**Remark 3.4.** Corollary 3.3 is a refinement of Theorem 1.2.

If we take $\mu = 1$ in Corollary 3.3, then we get

**Corollary 3.5.** *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(\alpha, \lambda)$. Then*

$$|a_2| \leq \begin{cases} \dfrac{2\alpha}{\lambda+1}, & \lambda \geq 1+\sqrt{2} \\[2ex] \sqrt{\dfrac{2}{2\lambda+1}}\alpha, & 1 \leq \lambda < 1+\sqrt{2} \end{cases}$$

*and*

$$|a_3| \leq \frac{2\alpha^2}{2\lambda+1}.$$

**Remark 3.6.** Corollary 3.5 provides an improvement of a result which was obtained by Frasin and Aouf [9, Theorem 2.2].

If we take $\lambda = 1$ in Corollary 3.5, then we have

**Corollary 3.7.** *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^\alpha(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2}{3}}\alpha,$$

*and*

$$|a_3| \leq \frac{2}{3}\alpha^2.$$

**Remark 3.8.** Corollary 3.7 provides a refinement of a result which was obtained by Srivastava [14, Theorem 1].

If we take $\lambda = 1$ and $\mu = 0$ in Corollary 3.3, then we get

**Corollary 3.9.** *Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^*[\alpha]$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \sqrt{2}\alpha,$$

*and*

$$|a_3| \leq 2\alpha^2.$$

**Remark 3.10.** Corollary 3.9 provides an improvement of estimates which was obtained by Brannan [3].

If we take $\delta = 1$ and $\mu = 1$ in Corollary 3.1, then we have

**Corollary 3.11.** *Let the function $f(z)$ given by (1.1) be in the class $B_{\Sigma}(n,\alpha,\lambda)$. Then*

$$|a_2| \leq \min\left\{ \frac{2\alpha}{2^n(\lambda+1)}, \frac{2\alpha}{\sqrt{2.3^n(2\lambda+1)}} \right\},$$

*and*

$$|a_3| \leq \frac{2\alpha^2}{3^n(2\lambda+1)}.$$

**Remark 3.12.** Corollary 3.11 provides a refinement of a result which was obtained by Porwal and Darus [12, Theorem 2.1].

By letting

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leq \beta < 1,\ z \in \mathbb{U}),$$

in Theorem 2.3, we deduce the following result.

**Corollary 3.13.** *Let the function $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\delta,\mu}(n,\beta,\lambda)$. Then*

$$|a_2| \leq \min\left\{ \frac{2(1-\beta)}{(1+\delta)^n(\lambda+\mu)}, \sqrt{\frac{4(1-\beta)}{|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}} \right\},$$

*and*

$$|a_3| \leq \min\{\frac{4(1-\beta)^2}{(1+\delta)^{2n}(\lambda+\mu)^2} + \frac{2(1-\beta)}{(1+2\delta)^n(2\lambda+\mu)}, \frac{(1-\beta)[|4(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|+(1+\delta)^{2n}|\mu-1|]}{(1+2\delta)^n|2(1+2\delta)^n+(1+\delta)^{2n}(\mu-1)|(2\lambda+\mu)}\}.$$

**Remark 3.14.** The results obtained in Corollary 3.13 are the same as the result which was obtained by Bulut [4, Theorem 12].

If we take $n = 0$ in Corollary 3.13, then we have

**Corollary 3.15.** *Let the function $f(z)$ given by (1.1) be in the class $N_{\Sigma}^{\mu}(\beta,\lambda)$. Then*

$$|a_2| \leq \min\left\{ \frac{2(1-\beta)}{(\lambda+\mu)}, \sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}} \right\},$$

*and*

$$|a_3| \leq \begin{cases} \min\{\frac{2(1-\beta)}{(2\lambda+\mu)} + \frac{4(1-\beta)^2}{(\lambda+\mu)^2}, \frac{4(1-\beta)}{(1+\mu)(2\lambda+\mu)}\}, & 0 \leqslant \mu < 1 \\ \frac{2(1-\beta)}{(\mu+2\lambda)}, & \mu \geq 1. \end{cases}$$

If we take $\mu = 1$ in Corollary 3.15, then we get

**Corollary 3.16.** *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(\beta,\lambda)$. Then*

$$|a_2| \le \min\left\{\frac{2(1-\beta)}{(\lambda+1)}, \sqrt{\frac{2(1-\beta)}{(2\lambda+1)}}\right\},$$

*and*

$$|a_3| \le \frac{2(1-\beta)}{(2\lambda+1)}.$$

**Remark 3.17.** Corollary 3.16 provides a refinement of a result which was obtained by Frasin and Aouf [9, Theorem 3.2].

If we take $\lambda = 1$ in Corollary 3.16, then we have

**Corollary 3.18.** *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)(0 \le \beta < 1)$. Then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & 0 \le \beta \le \frac{1}{3} \\ (1-\beta), & \frac{1}{3} \le \beta < 1 \end{cases}$$

*and*

$$|a_3| \le \frac{2(1-\beta)}{3}.$$

**Remark 3.19.** Corollary 3.18 provides an improvement of a result which was obtained by Srivastava [14, Theorem 2].

If we take $\lambda = 1$ and $\mu = 0$ in Corollary 3.15, then we get

**Corollary 3.20.** *Let the function $f(z)$ given by (1.1) be in the class $S^*_\Sigma[\beta]$ $(0 \le \beta < 1)$. Then*

$$|a_2| \le \begin{cases} \sqrt{2(1-\beta)}, & 0 \le \beta \le \frac{1}{2} \\ 2(1-\beta), & \frac{1}{2} \le \beta < 1 \end{cases}$$

*and*

$$|a_3| \le \begin{cases} 2(1-\beta), & 0 \le \beta \le \frac{3}{4} \\ (1-\beta)(5-4\beta), & \frac{3}{4} \le \beta < 1. \end{cases}$$

**Remark 3.21.** Corollary 3.20 provides a refinement of estimates which was obtained by Brannan [3].

If we take $\delta = 1$ and $\mu = 1$ in Corollary 3.13, then we have

**Corollary 3.22.** *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(n,\beta,\lambda)$. Then*

$$|a_2| \le \min\left\{\frac{2(1-\beta)}{2^n(\lambda+1)}, \sqrt{\frac{2(1-\beta)}{3^n(2\lambda+1)}}\right\},$$

*and*

$$|a_3| \leq \frac{2(1-\beta)}{3^n(2\lambda+1)}.$$

**Remark 3.23.** Corollary 3.22 provides an improvement of a result which was obtained by Porwal and Darus [12, Theorem 3.1].

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